SAMPLE

Math 5B Test 3
Spring 2013
(11.8-11.11, 8.1-8.3)

100 POINTS **NAME:** Solns

No scratch paper. Show all work clearly on test paper. No credit will be given for solutions if work is not shown. Only non-graphing calculators are allowed. Unless otherwise specified, the answer to series questions should be given using sigma notation. Unless otherwise stated, you do not need to find the radius of convergence.

(1) FIND THE INTERVAL OF CONVERGENCE FOR EACH OF THE FOLLOWING. (5 points each) Use Ratio Test.

Need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

Check endpts where =

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{\sqrt{n}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^{n+1}(x+3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{2^n(x+3)^n}\right|$$

$$\Rightarrow \lim_{n\to\infty} 2|x+3|\frac{\sqrt{n}}{\sqrt{n+1}} = 2|x+3|\cdot 1$$

Series converges absolutely when

$2|x+3| < 1$

$\Rightarrow |x+3| < 1/2 \Rightarrow -\frac{7}{2} < x < -\frac{5}{2}$

Check endpoints

$x = -\frac{7}{2}$ $\displaystyle\sum_{n=0} \frac{2^n(-\frac{1}{2})^n}{\sqrt{n}} = \sum_{n=0} \frac{(-1)^n}{\sqrt{n}}$ conv. by A[illegible]

$x = -\frac{5}{2}$ $\displaystyle\sum_{n=0} \frac{2^n(\frac{1}{2})^n}{\sqrt{n}} = \sum_{n=0} \frac{1}{\sqrt{n}}$ div $p = 1/2$

Answer: $\boxed{[-7/2, -5/2)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2\cdot 4\cdot 6\cdot\ldots\cdot(2n)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^{n+1}}{2\cdot4\cdot6\cdot\cdots\cdot 2(n+1)} \cdot \frac{2\cdot4\cdot\ldots\cdot 2n}{n^2 x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2 |x|}{(2n+2)\,n^2}\right| = 0 \text{ for all } x.$$

$(-\infty, \infty)$

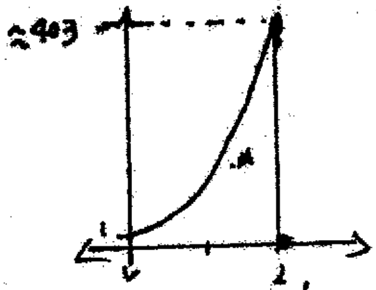
(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}(x+4)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(n+1)!(x+4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n!(x+4)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{3}|x+4| = \infty \text{ unless } x = -4$$

Converges for $x = -4$ only

(2) Find the centroid of the region bounded by the graphs of $y = e^{3x}$, $y=0$, $x=0$, and $x=2$. (10 points)



$$\bar{x} = \frac{M_y}{m} = \frac{\rho\left(\frac{5}{9}e^6 + \frac{1}{9}\right)}{\frac{\rho}{3}(e^6-1)} = \frac{5e^6+1}{3e^6-3} \approx 1.67$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{\rho}{12}(e^{12}-1)}{\frac{\rho}{3}(e^6-1)} = \frac{e^{12}-1}{4e^6-4} \approx 101.1$$

$$\text{mass} = m = \int_a^b \rho f(x)\,dx = \int_0^2 \rho e^{3x}\,dx = \frac{\rho}{3}e^{3x}\Big]_0^2 = \frac{\rho}{3}(e^6-1)$$

$$\text{Moment about } y = M_y = \int_a^b \rho x f(x)\,dx = \int_0^2 \rho x e^{3x}\,dx$$ BY parts $u=x$, $du=dx$, $dv=e^{3x}dx$, $v=\frac{1}{3}e^{3x}$

$$= \rho\left(\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_0^2 = \underline{\rho\left(\frac{5}{9}e^6 + \frac{1}{9}\right)}$$

$$\text{Moment about } x = M_x = \int_a^b \rho \frac{1}{2}(f(x))^2\,dx = \int_0^2 \frac{\rho}{2}(e^{3x})^2\,dx = \frac{\rho}{2}\int_0^2 e^{6x}\,dx = \underline{\frac{\rho}{12}(e^{12}-1)}$$

(3) Find the Maclaurin series for $f(x)=\cos 2x$ directly, using the definition. – know know series and no substitution (10 points)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Find $f^{(n)}(0)$

$f(x) = \cos 2x \qquad f(0) = 1$

$f'(x) = -\sin 2x \cdot 2 \qquad f'(0) = 0$

$f''(x) = -\cos 2x \cdot 2^2, \qquad f''(0) = -2^2$

$f'''(x) = \sin 2x \cdot 2^3 \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos 2x \cdot 2^4 \qquad f^{(4)}(0) = 2^4$

Cannot write general term for $f^{(n)}(0)$ easily so write some terms

$$\cos 2x = 1 + 0x - \frac{2^2}{2!}x^2 + 0x^3 + \frac{2^4}{4!}x^4 + \cdots$$

$$\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 + \cdots$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{2^{2n}x^{2n}}{(2n)!}\cdot(-1)^n$$

Notice: easy to check using know formula $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

And substitute $2x$ for x.

(4) Find the Maclaurin series for $x^4 e^{x^3}$ (5 points)

(There are easy ways and there are hard ways this can be done)

Know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

substitute x^3 for x

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \text{ for all } x$$

Mult. by x^4

$$x^4 e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \cdot x^4 = \underline{\underline{\sum_{n=0}^{\infty} \frac{x^{3n+4}}{n!}}} \text{ for all } x$$

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(5) Find the Taylor series for $f(x)=1/x^2$ centered at a=2. (Assume that f has a power series expansion.) (10 points)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

Find $f^{(n)}(2)$

$f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$f''(x) = 3 \cdot 2x^{-4}$

$f'''(x) = -4 \cdot 3 \cdot 2x^{-5}$

$\vdots$

$f^{(n)}(x) = (-1)^n (n+1)! \, x^{-(n+2)}$

so

$$f^{(n)}(2) = \frac{(-1)^n (n+1)!}{2^{n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{2^{n+2}\, n!}(x-2)^n$$

$$= \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}}(x-2)^n}}$$

(6) Find the length of the curve $y = x^{2/3}$ from $(1,1)$ to $(2\sqrt{2}, 2)$

(10 points)

$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ OR $L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$y = x^{2/3}$ $y' = \frac{2}{3x^{1/3}}$

$x = y^{3/2}$ $x' = \frac{3}{2} y^{1/2}$

$L = \int_1^{2\sqrt{2}} \left(1 + \frac{4}{9x^{2/3}}\right) dx = \int_1^{2\sqrt{2}} \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\, dx$

$L = \int_1^2 \sqrt{1 + \frac{9}{4} y}\, dy$ $u = 1 + \frac{9}{4} y$

$du = \frac{9}{4} dy$

$u = 9x^{2/3} + 4$

$du = 6x^{-1/3} dx$

$\frac{4}{9} \int_{13/4}^{11/2} u^{1/2} du$

$\frac{1}{18} \int_{13}^{22} u^{1/2} du = \frac{1}{27} u^{3/2} \Big]_{13}^{22} = \frac{1}{27} \left(22^{3/2} - 13^{3/2}\right)$

$\frac{8}{27} u^{3/2} \Big]_{13/4}^{11/2} = \frac{8}{27} \left(\left(\frac{11}{2}\right)^{3/2} - \left(\frac{11}{4}\right)^{3/2}\right)$

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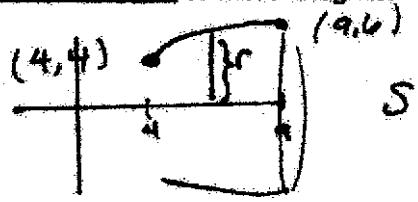
(7) Find the area of the surface generated by rotating the curve $y = 2\sqrt{x}$, $4 \le x \le 9$ about the x axis.

Set up two different integrals,

(a) by integrating with respect to x, and

(b) by integrating with respect to y.

Evaluate either one of these integrals.



$S = \int 2\pi r\, ds$

(10 points)

a) $\int_4^9 2\pi \, 2\sqrt{x} \sqrt{1 + \frac{1}{x}}\, dx$

b) $\int_4^6 2\pi \, y \sqrt{1 + \left(\frac{y}{2}\right)^2}\, dy$

Ans: $\frac{8\pi}{3} \left(10^{3/2} - 5^{3/2}\right)$

(8) Using the geometric series for $\frac{1}{1-x}$ find a power series representation for $\frac{3x}{1+2x}$ and $\frac{5x}{1+3x}$ determine the radius of convergence.

(10 points)

$$\frac{5x}{1+3x} = 5x \cdot \frac{1}{1-(-3x)}$$

substitute $-3x$ for x

$$\frac{1}{1+3x} = \sum_{n=0} (-3x)^n \quad |3x|<1 \Rightarrow |x|<$$

mult by $5x$

$$\frac{5x}{1+3x} = 5\sum_{n=0}^{\infty} (-1)^n 3^n x^{n+1} \quad |x|<1/3$$

radius

(9) Use series to compute $\int_0^{1/2} x^2 e^{-x^2}\,dx$ with $|\text{error}| < 0.001$ (10 points)

(with minimal # terms)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

$$e^{-x^2} = \sum_{n=0} \frac{(-x^2)^n}{n!} = \sum_{n=0} \frac{(-1)^n x^{2n}}{n!} \text{ for all } x$$

$$x^2 e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} \quad \text{for all } x$$

$$\int_0^{1/2} x^2 e^{-x^2} dx = \int_0^{1/2} \sum_{n=0} \frac{(-1)^n x^{2n+2}}{n!} dx = \sum_{n=0} \frac{(-1)^n x^{2n+3}}{(2n+3)\,n!} \Bigg]_0^{1/2} \text{ for all } x$$

$$= \sum_{n=0} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+3}}{(2n+3)\,n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3}(2n+3)\,n!}$$

/5

$$= \underbrace{\frac{1}{2^3 \cdot 3} - \frac{1}{2^5 \cdot 5}} + \frac{1}{2^7 \cdot 7 \cdot 2!} - \frac{1}{2^9 \cdot 9 \cdot 3!}$$

.00055
Smaller than .001

$$\approx \frac{1}{24} - \frac{1}{160}$$

$$\approx .035$$

(10) 5 (a) Approximate the function $f(x) = x\ln x$ by $T_3(x)$, the third degree Taylor Polynomial centered at a=1.

3 (b) Use Taylor's inequality to estimate the accuracy of the approximation when x lies in the interval $0.9 \le x \le 1.1$

2 (c) Use $T_3(x)$ to approximate (1.01) ln(1.01)

(10 points)

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$f(x) = x\ln x \qquad f(1) = 0$

$f'(x) = 1 + \ln x \qquad f'(1) = 1$

$f''(x) = \frac{1}{x} \qquad f''(1) = 1$

$f'''(x) = \frac{-1}{x^2} \qquad f'''(1) = -1$

$f^{(4)}(x) = \frac{2}{x^3}$

a) $x\ln x \approx (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{3!}$

b) $|R_3(x)| \le \frac{f^{(4)}(c)\,(x-1)^4}{4!}$ for c between x and a=1

Book's "M" $= \frac{2}{c^3}\frac{(x-1)^4}{24}$ $\qquad 0.9 \le x \le 1.1 \Rightarrow |x-1| \le 0.1$

$= \frac{2}{c^3}\frac{(0.1)^4}{24}$ This is largest when c is small

.9 1 1.1

$\le \frac{2\,(0.1)^4}{(.9)^3\,24} \approx 0.00001$

c) $1.01\ln(1.01) \approx T_3(1.01)$

$= (1.01-1) + \frac{(1.01-1)^2}{2} - \frac{(1.01-1)^3}{6}$

$= .01 + \frac{(.01)^2}{2} - \frac{(.01)^3}{6} \qquad .010049833$

(Calculator: 0.010049833...)

double check should be at least within 0.0000