Comparing the rank (A) to the rank [A|b]

For the matrices A and b below, find the rank of A and the rank of the augmented matrix [A|b]

<u>Example 1:</u> $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

To find the rank of A we will row reduce A and find the number of vectors in a basis of the row space of A.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ So the rank(A)=3

Similarly, if we row reduce the augmented matrix [A|b]

$$[A \vdots \vec{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ we find the rank[A|b] = 3.

In this example, the system $A\vec{x} = \vec{b}$ has a unique solution.

<u>Example 2:</u> $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ rank(A) = 2

$$[A \vdots \vec{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ rank[A|b] = 2

The system $A\vec{x} = \vec{b}$ has infinitely many solutions.

<u>Example 3:</u> $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ rank(A) = 2

$$[A \vdots \vec{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$ rank[A|b] = 3

The system $A\vec{x} = \vec{b}$ has no solution

Generalizing….

Theorem: Let $A\vec{x} = \vec{b}$ be a system of linear equations in n variables.

1) If rank(A)=. rank[A|b] = n then the system has a unique solution, (consistent)
2) If rank(A)=. rank[A|b] < n then the system has infinitely many solutions, (consistent)
3) If rank(A)<. rank[A|b] then the system has no solution,

So for a given A and b, the system $A\vec{x} = \vec{b}$ is consistent if and only if rank(A)=. rank[A|b]