Examples: Nullspace, Row Space & Column Space

Example 1: Given the 3X4 matrix $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ Find:

a) The solution(s) to the system $Ax = 0$

b) A basis for the nullspace of $A$, and its dimension.

c) A basis for the row space of $A$, and its dimension.

d) A basis for the column space of $A$, and its dimension.

Solution:

a) The solution(s) to the system $Ax = 0$

The augmented matrix for the system, $\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{bmatrix}$ can be row reduced to $\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which yields the solution $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - \frac{2}{7} t \\ -s - \frac{4}{7} t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$.

Notice there are two parameters, $s$ & $t$. $x_3$ and $x_4$ are called free variables, $x_1$ and $x_2$ are called leading variables. So in this example there are two free variables and two leading variables.

b) A basis for the nullspace of $A$, and its dimension.

Using part (a), a basis for the nullspace of $A$ is $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Since there are two vectors in this basis, the dimension of the nullspace of $A$ is 2. This is called the nullity. $\text{Nullity}(A)=2$.

c) A basis for the row space of $A$, and its dimension.

Row reducing… $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So a basis is $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \end{bmatrix}$ and the dimension of the row space of $A$ is 2.

d) A basis for the column space of $A$, and its dimension.

Two ways you can do this.

1) Using the row echelon form of $A$ in part (c) we can see that the first two columns form a basis for the column space of that matrix, thus the first two columns of $A$ form a basis for the column space of $A$. So a basis is $\begin{bmatrix} 1 & 4 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}$ and the dimension of the column space of $A$ is 2.

2) Since the column space of $A = $ the row space of $A^T$ we can find a basis for the row space of $A^T$. 

\[ A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]. Now since \( \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \) is a basis for the row space of \( A^T \), \( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \) is a basis for the column space of \( A \).
Example 2: Given the 2X2 matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ Find:

a) The solution(s) to the system $A\vec{x} = \vec{0}$

b) A basis for the nullspace of $A$, and its dimension.

c) A basis for the row space of $A$, and its dimension.

d) A basis for the column space of $A$, and its dimension.

Solution:

a) The solution(s) to the system $A\vec{x} = \vec{0}$

The augmented matrix for the system, $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \end{bmatrix}$ can be row reduced to $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which yields the solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We should have expected this since the $\det(A) \neq 0$, so $A$ is invertible and $A\vec{x} = \vec{0}$ has only the trivial solution.

b) A basis for the nullspace of $A$, and its dimension.

Using part (a), the nullspace is just the zero space so there is no basis and the dimension is zero. $\text{Nullity}(A)=0$.

c) A basis for the row space of $A$, and its dimension.

Row reducing... $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. So a basis is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and the dimension of the row space of $A$ is 2. Notice, in this case, since it is easy to see that the rows of $A$ are two linearly independent vectors, the row space is $\mathbb{R}^2$ so any two linearly independent vectors will form a basis.

d) A basis for the column space of $A$, and its dimension.

Using the row echelon form of $A$ in part (c) we can see that the two columns form a basis for the column space of that matrix, thus the two columns of $A$ form a basis for the column space of $A$. So a basis is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$ and the dimension of the column space of $A$ is 2.
Example 3:

(a) Find a basis for the row space of \( A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \) that consists entirely of row vectors of \( A \).

(b) Find a subset of the vectors \( v_1 = (1, 4, 5, 2), v_2 = (2, 1, 3, 0), v_3 = (-1, 3, 2, 2) \) that forms a basis for the space spanned by these three vectors.

(c) Express each vector not in the basis as a linear combination of the basis vectors.

Solution:

(a) Since row space of \( A = \) column space of \( A^T \), we'll find the column space of \( A \) using the first method in example 1d.

\[
A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Using the row echelon form of \( A^T \) we can see that the first two columns form a basis for the column space of \( A^T \). Thus \( \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix} \) form a basis for the row space of \( A \).

(b) This is the same problem, other than notation. So \( v_1 = (1, 4, 5, 2) \) & \( v_2 = (2, 1, 3, 0) \) form a basis for the space spanned by the three given vectors.

(c) Since \( v_1 \) & \( v_2 \) form a basis for the space spanned by the three given vectors, \( v_3 \) can be written as a linear combination of \( v_1 \) & \( v_2 \). This combination is most easily found by continuing the row operations from part (a) to obtain reduced row echelon form.

\[
A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

From here we can see that the third column is one times the first column – one times the second which holds true in \( A^T \) as well. Since the columns of \( A^T \) are precisely the given vectors, we have \( v_3 = v_1 - v_2 \).